Using Gamification to Engage Students While Learning Mathematical Induction

Tim Gegg-Harrison and Nicole Anderson
Department of Computer Science
Winona State University
Winona, MN 55987
{tgeggharrison,nanderson}@winona.edu

Abstract

While students are often intimidated by discrete mathematics concepts such as proof by induction, they are often drawn to games and specifically games on mobile devices. The turn-based nature of many mobile apps also provides a social connection with their classmates, allowing students an interactive learning experience. We have constructed a mobile app to be used concurrently with the presentation of the concept of mathematical induction in a discrete mathematics course. The app allows students to play a turn-based strategy game that captures the subtraction game variant of *Nim*, where players take turns removing objects from a pile and the player that removes the last object loses. Strategy is clearly important and devising an optimal strategy is possible and can be proved using mathematical induction. Our objective is to use the game side-by-side with a proof by induction to introduce the concept of mathematical induction in a fun and engaging manner.

1 Introduction

For students, discrete mathematics is a course that can be challenging and may be seen as a subject that they must "get over with" in order to study other subjects in computer science. This is especially true of the concept of mathematical induction introduced in the discrete mathematic course. In this work, we attempt to use gamification to both help students understand the concept of mathematical induction more completely, as well as to make the learning of the topic more fun and engaging.

Gamification is the use of game elements to improve user experience and user engagement, specifically in non-game contexts [2,3]. It has been used in many contexts, from military training [1,6] to education [10] to selling consumer goods [13]. "By 2015, over 50% of organizations are predicted to be using gamification – and by 2016 it is estimated that $2.8 billion will be spent on it" [7]. Worldwide, people spend 3 billion hours a week playing video and computer games [8]. Wouldn't it be nice to direct our students' time toward understanding computer science concepts instead of perfecting their skill launching Angry Birds?

According to Ziebermann, Gabe, and Cunningham , "when done well, gamification helps align our interests with the intrinsic motivations of our players, amplified with the mechanics and rewards that make them come in, bring friends, and keep coming back" [14]. This is precisely our goal in this work. We have created an iPad app called *Disperse* which is loosely based on the subtraction game variant of *Nim*. The *Disperse* game provides a fun interactive example that helps students learn mathematical induction. Since the game is turn-based, it provides a social atmosphere conducive to learning. In addition, the "rewards" the students get by advancing up the leaderboard provide motivation for the students to perfect their game play (i.e., develop an optimum strategy that they can then prove using mathematical induction).

Gamification is not without its detractors. Some argue that "pointsification" is all that is achieved by many so-called "gamified" applications [12]. We believe our app goes beyond this and provides a new, additional method for students to learn mathematical induction beyond the traditional teaching techniques. Even if gamification does not provide a one-size-fits-all mechanism of learning these concepts, research shows that supporting multiple intelligences and learning styles to gain concepts is optimal [5,4].

2 Using Gamification to Learn Mathematical Induction

In this section, we introduce the game of *Nim* as well as the *Disperse* app, which is loosely based upon the subtraction variant of *Nim*. These examples will focus on finding an optimal strategy for these games, and then proving the optimal strategy using proof by induction. We also examine how using gamification to teach mathematical induction engages the students differently than traditional methods of teaching applied in a discrete mathematics course.

2.1 *Nim*

Nim is a game example that may be used to illustrate mathematical concepts such as commutative algebra [11] and mathematical induction. It works well as the game itself is easy to learn so the focus can be placed on the new mathematical concepts.

2.1.1 Classic *Nim*

The classic *Nim* game uses heap(s) of objects, often represented as a pile (or set of piles) of sticks. It is played by two players who take turns removing one or more objects from one or more of the heaps. The *Nim* game has several variations. In some variations taking the last object means you are the loser of the game, in others it makes you the winner. Some use a single heap while others use multiple heaps for gameplay. The specific version of the *Nim* game that we use for our purposes is the subtraction game variant.

2.1.2 Subtraction Game Variant of *Nim*

In the subtraction game variant of *Nim*, a limit is placed on the number of objects that may be removed from the heap during each turn. It is played as a misère game in that the player to take the final object loses the game.

2.2 *Disperse*

We have built upon the subtraction game variant of *Nim* to create the *Disperse* game and realize it as an iPad app [9]. In *Disperse* the objects on our pile are playing cards. It is a turn-based game using the subtraction game variant of *Nim* where the player to take the last card loses and the limit on the number of cards a player may take on a given turn is four. However, with this game there is an additional twist. Players can pick up and most one card of each suit, and can only pick up fully exposed cards. That is, if a card is partially covered by another card it may not be selected and removed. Thus, a winning strategy for *Disperse* is more complex than what is required for the subtraction game variant of the *Nim* game.

2.3 An Exercise for Learning Mathematical Induction using *Disperse*

We have created a three part exercise that introduces students to mathematical induction through the *Disperse* iPad app. Prior to being given the exercise, students are given a brief introduction to mathematical induction through an example of two individuals who throw a party and challenge one another to see if it possible for everyone attending the party to shake a different number of hands during the evening. This is followed by some examples of mathematical induction to prove the completeness of various number representations. In the first part of the exercise, students are asked to download *Disperse* onto their iPads and play 6 or more games with their classmates. After playing these games, students are asked to think about effective winning strategies that they employed. In the second part of the exercise, students are asked to consider the less sophisticated subtraction variant of *Nim* and are guided through the identification of the optimal strategy for winning whenever possible. In the third and final part of the exercise, students are given the optimal strategy for playing the subtraction variant of *Nim* along with the beginning of a proof of its optimality using (strong) mathematical induction. The proof that the students are given is flawed because it does not have sufficient basis cases. Students are asked to identify and fix the flaw and then complete the proof.

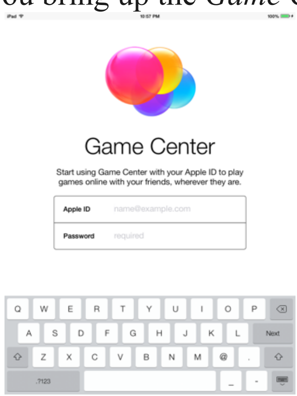
2.3.1 Thinking about Winning Strategies for *Disperse*

The first part of the exercise is done outside of class. In this part of the exercise, students are asked to set up Game Center (if they currently do not have an account with Game Center), download the *Disperse* app with their classmates and/or friends, and to reflect on the effective strategies that were employed.

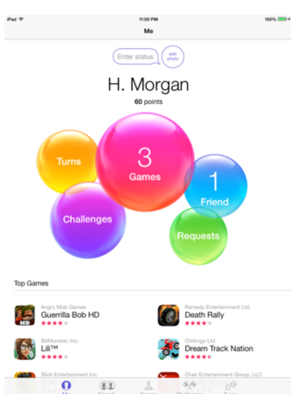
<u>Exercise 1</u>

1. Set up Game Center

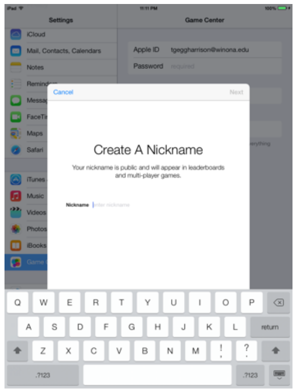
   There are two ways to set up Game Center. One is by bringing up the *Game Center* app and logging in using your Game Center account. Note that although your Game Center account uses your Apple ID, it requires additional information. Specifically, it requires a nickname that can be used to identify you on leaderboards. If you are not already logged in, you will see the following screen when you bring up the *Game Center* app:



   After logging in (or you were already logged in), you will see the following screen, which allows you access to your active games, friends, etc.:



The other way to set up Game Center is through the *Settings* app. Select Game Center and then enter an Apple ID and password if you are not already logged in. If it is your first time logging in then you will see the following screen to create a nickname (which must be unique systemwide):



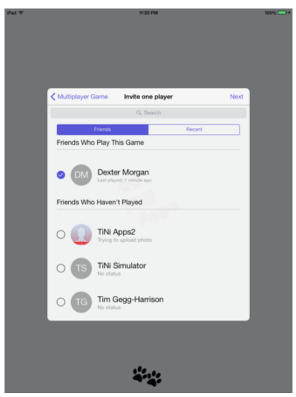
Note that you can log out of Game Center if you need to sign in with a different Apple ID (if you want to let a friend play on your iPad or if you have multiple personalities) by selecting your Apple ID and then selecting Sign Out.

2. Download *Disperse*.

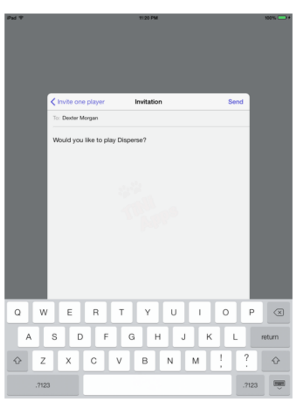
   You can download the *Disperse* app to your iPad by bringing up the *App Store* app on your iPad and searching for *tiniapps*. *Disperse* should be one of the first few apps to show up. You can also find it by going to http://www.tiniapps.com/ and clicking on the *Disperse* icon. This will take you to the App Store where you can download the *Disperse* app and then transfer it to your iPad. Once it is downloaded to your iPad, run it. Wait until you see the "Welcome back" message at the top. If you miss it, you'll know it happens if the light gray box behind the "HOW TO PLAY" rules has "Multiplayer Game" on it:

   The rules of the game are fairly simple. You can remove between one and four cards (up to one card of each suit) on your turn as long as each of the cards that you want to remove is not covered by another card. To help you out, *Disperse* highlights the cards which you may select, and the available suits are in the corners of the screen. On your turn, simply slide the cards you wish to remove off the screen, one at a time. You win the round by forcing your opponent to remove the last card on the pile, and you win the game by winning three rounds before your opponent.

Click the *I'm ready to play!* button and you will be asked to *Invite Friend*. Don't hit the *Play Now* bubble because that is intended to create an auto-match game (which is currently not supported).



Find the friend that you want to play, click the circle to left of their name, and then press *Next* in the upper lefthand corner of the screen. In this case, we have decided to play Dexter Morgan.



Simply press *Send* in the upper righthand corner of the screen and the the game will begin. You will briefly see the following screen:

And then you will see a pile of cards with the ones on top highlighted green. At this point, it is your turn so you can drag any of the highlight cards off the screen (one at a time). Notice that the only exposed cards are the King of Diamonds and Jack of Spades. As such, they are highlighted.

The following screen captures the player in the process of moving the Jack of Spades off the lefthand side of the screen. You can slide them any direction to remove them.

Notice that the Queen of Diamonds was exposed when the Jack of Spades was removed so it is also highlighted now. Also notice that the spade in the upper righthand corner of the screen is gone (indicating that you can no longer remove a spade).

Assume the player removes the Queen of Diamonds, exposing the Ace of Clubs.

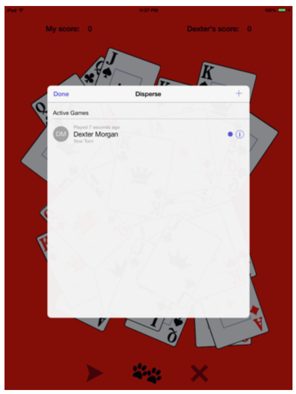
Notice how the King of Diamonds is no longer highlighted (since the player just removed a diamond) and that the Ace of Clubs is now highlighted (since it is no longer covered by the Queen of Diamonds that the player removed). Also notice that the diamond in the lower lefthand corner is now gone as well. In order to complete you turn, you must press the play button in the bottom left. Note that you press the play button anytime after you have removed a card. You have to remove at least one card, but you do not have to remove cards just because you can. For this example though, let's assume the player continues to remove cards and removes the Ace of Clubs.

Now the King of Hearts of exposed and therefore highlighted. Assume the player removes the King of Hearts.

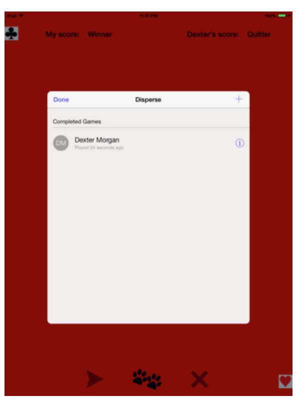
Now, the player is not able to remove anymore cards since one of each suit has been removed. At this point, the player presses the play button. After the player presses the play button, the screen will become red (to indicate that it is not your turn). Notice also that the play button is disabled. You can still press the other buttons at the bottom. The X on the right will allow you to quit the game. The other one with the 2 paws will take you to Game Center.

From Game Center, you will be able to see all of your active games. In our case, we only have one game so it is all that shows. Pressing on your opponent's name will bring you back into the game.

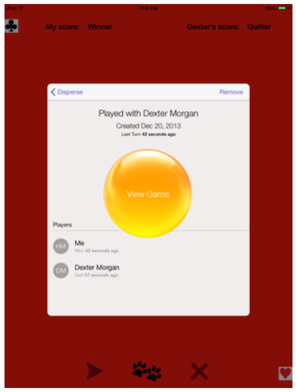
In the following screen, it is our turn again (as indicated by the blue dot after our opponent's name).



When the game is over, we can still see it in Game Center.



Clicking the circled i will give us the following screen where we can remove the game from Game Center by pressing the *Remove* button in the upper righthand corner of the screen.



3. Play *Disperse*

   Now, make some friends by going to *Game Center* and clicking on the *Friend* bubble (or selecting the second tab from the right at the bottom of the screen). Press the + in the upper righthand corner and send a friend request. Note that you have to befriend someone before you can play them. Play at least 6 complete games (which consist of 3 out of 5 rounds). There's a leader board for *Disperse* so see if you can climb it.

4. Reflect on game strategies

   After playing 6 games of *Disperse*, take a moment to reflect on how you played the game. What strategies did you use? Were they successful?

2.3.2 Identifying the Optimal Strategy for the Subtraction Game Variant of *Nim*

The second part of the exercise is done in class in groups of two or three. In this part of the exercise, students are guided through the construction of an optimal strategy for playing the subtraction game variant of *Nim*.

Exercise 2 Discussion

> As you played *Disperse* for Exercise 1, you developed strategies that helped you win the game more frequently. However, the ordering of the cards on the pile limited the way that you could play and made defining an optimal strategy more challenging. *Disperse* is based on a children's game that is played with sticks that is called *Nim*. The subtraction game variant is simple. A pile of sticks is placed in front of a pair of children. The constraint that allowed players to take away only one card of each suit per turn is removed. Instead, the children take turns removing either 1, 2, 3, or 4 sticks from the pile. The child who removes that last stick from the pile loses the game. It turns out that like many children's games, there is an optimal strategy for playing this subtraction game variant of *Nim*. We can actually use mathematical induction to construct this optimal strategy.
>
> We begin by considering the rules of the game. A player loses the game if he/she is forced to pick up the last stick in the pile. Thus, a pile containing a single stick

is bad pile. Other piles of sticks are not so bad. Consider a pile that contains 2 sticks. If it is your turn and you have a pile with 2 sticks then you can pick up a single stick which will leave your opponent with a bad pile containing a single stick. Likewise, if it is your turn and you have a pile with 3 sticks then you can pick up 2 sticks which will leave your opponent with a bad pile containing a single stick. And if it is your turn and you have a pile with 4 sticks then you can pick up 3 sticks which will leave your opponent with a bad pile containing a single stick. Finally, if it is your turn and you have a pile with 5 sticks then you can pick up 4 sticks which will leave your opponent with a bad pile containing a single stick.

Note that if it is your turn and you a have a pile with 6 sticks then there is nothing you can do to prevent your opponent from giving you a bad pile after his/her turn. If you take a single stick then he/she can take 4 sticks, leaving you with a bad pile. If, on the other hand, you take 2 sticks then he/she can take 3 sticks, leaving you with a bad pile. If your take 3 sticks then he/she can take 2 sticks, leaving you with a bad pile. Finally, if you take 4 sticks then he/she can take a single stick, leaving you with a bad pile. So, a pile with 6 sticks is just as bad as a pile with a single stick.

A pile with 7 sticks, on the other hand, is great because you can take a single stick and force your opponent to have to deal with a bad pile containing 6 sticks. Likewise, you can force your opponent to have to deal with a bad pile containing 6 sticks if you have a pile with 8, 9, or 10 sticks by removing 2, 3, or 4 sticks, respectively. A pattern is clearly arising.

Exercise 2 Questions

1) Identify the pattern. For a pile containing *n* sticks, which ones are good? Which ones are bad? Express the pattern in the most general way possible.

2) Remember that you have no control over how many sticks your opponent is going to remove, so you must consider all possibilities. Make a claim that expresses the optimal strategy for the subtraction game variant of *Nim*. For each of the good piles covered by the pattern you identified in the previous problem, how many sticks should you remove so that you can ensure that you will win?

2.3.3 Verifying the Optimal Strategy for the Subtraction Game Variant of *Nim*

The third part of the exercise is done in class immediately following Exercise 2 with the same groups of two or three. In this part of the exercise, students are presented with the initial part of failed proof of the optimality of the strategy that they constructed in Exercise 2. Note that the complete proof contains 20 subcases. As such, we usually ask our students to produce only the case when $k + 1 = 5m$ or $5m + 1$ or $5m + 2$.

Exercise 3 Discussion

Examine the following failed proof for our claim about the optimal strategy for playing the subtraction game variant of *Nim*:

Claim: It is possible to guarantee a win in the subtraction game variant of *Nim* as long as the number of sticks in the pile is not $5m + 1$ for some $m \geq 0$.

Proof Attempt
Let $P(n)$: It is possible to guarantee a win in the subtraction game variant of *Nim* if the pile contains *n* sticks as long as $n \neq 5m + 1$ for all $m \geq 0$.

Basis case, for $n = 1,2,3,4,5$

$P(1)$*:* Since $1 = 5(0) + 1$, we do not have to guarantee a win (which we cannot) in this case. Therefore $P(1)$ is true.

$P(2)$: By removing a single stick, our opponent will be forced to remove the last stick from the pile thereby guaranteeing a win. Therefore $P(2)$ is true.

$P(3)$: By removing 2 sticks, our opponent will be forced to remove the last stick from the pile thereby guaranteeing a win. Therefore $P(3)$ is true.

$P(4)$: By removing 3 sticks, our opponent will be forced to remove the last stick from the pile thereby guaranteeing a win. Therefore $P(4)$ is true.

$P(5)$: By removing 4 sticks, our opponent will be forced to remove the last stick from the pile thereby guaranteeing a win. Therefore $P(5)$ is true.

Inductive case

Assume $P(i)$ is true for all *i* with $1 \leq i \leq k$: It is possible to guarantee a win in the subtraction game variant of *Nim* if the pile contains *i* sticks as long as $n \neq 5m + 1$ for all $m \geq 0$.

We must show $P(k + 1)$: It is possible to guarantee a win in the subtraction game variant of *Nim* if the pile contains $k + 1$ sticks as long as $k + 1 \neq 5m + 1$ for all $m \geq 0$, for $k + 1 \geq 6$. Now, we have 5 cases to consider with $k + 1$. Either $k + 1 = 5m$ or $k + 1 = 5m + 1$ or $k + 1 = 5m + 2$ or $k + 1 = 5m + 3$ or $k + 1 = 5m + 4$.
Let's consider the first case, when $k + 1 = 5m$. If $k + 1 = 5m$ then by removing 4 sticks, our opponent will be left with a pile containing $k - 3 = 5m - 4$ sticks. He/she has 4 possible moves. We must consider each of these moves. If our opponent removes a single stick then we will

have a pile containing $k-4=5m-5=5(m-1)$ sticks. Since $k+1 \geq 6$, it follows that $k-5 \geq 1$, so we know that $1 \leq k-4 \leq k$. Since $1 \leq k-4 \leq k$, we know by the inductive hypothesis that $P(k-4)$ is true. And since $k-4=5m-5=5(m-1)$, we know that it is possible to guarantee a win in this case. It is possible, however, that our opponent removes 2 sticks on his/her turn. In this case, we will have a pile containing $k-5=5m-6=5(m-2)+4$ sticks. Since $k+1 \geq 6$, it follows that $k-5 \geq 0$, so we know that $0 \leq k-5 \leq k$. Since $0 \leq k-5 \leq k$, we know by the inductive hypothesis that $P(k-5)$ is true.

Exercise 3 Questions

1) We can't say that $P(k-5)$ is true. Why not? Explain the problem with this claim. How can we fix it?

2) Develop a valid proof by induction for the strategy you developed for our simplified game. Show all of your work.

3 Conclusion

Discrete mathematics is foundational to computer science (CS) students. Understanding mathematical induction, in particular, is essential for all of our CS students. Unfortunately, because they lacked motivation and did not see the relevance of most discrete mathematics topics, in prior years our CS students were not performing well in their discrete mathematics class and were not retaining the material.

In order to help our students see the relevance of discrete mathematics, we created an algorithms-based discrete mathematics course. We attempted to get the students more actively engaged by turning the homework assignments into social events, where we encouraged students to work through the homework assignments in groups. Rather than having them turn in their solutions to these problems, they were given a 10-15 minute (individual) quiz at the beginning of the each class period that consists of a subset of the homework problems. We saw a drastic improvement in their performance and their retention of the material. However, some of our students were still viewing the course as something they had to "get over with" in order to get to the interesting concepts of computer science.

In this paper, we presented our most recent attempt to provide an active learning environment for our discrete mathematics students. One of the biggest hurdles for CS students learning complex discrete mathematics topics is that the real significance (from a CS perspective) of the topic is not truly appreciated until it is used in a follow-on CS class like data structures. In essence, we are forcing our students to delay satisfaction of the topic until a future class where they see its relevance. By *gamifying* the study of complex topics like mathematical induction, we are able to motivate our students and give them a reason for learning the topic when it is presented without the need to wait to see the relevance.

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